

Like Radicals are radical expressions that have the same index and radicand

**Combining Radical Expressions: Sums and Differences**

Use the distributive property to add or subtract like radicals

$$a\sqrt[n]{x} + b\sqrt[n]{x} = (a+b)\sqrt[n]{x}$$

$$a\sqrt[n]{x} - b\sqrt[n]{x} = (a-b)\sqrt[n]{x}$$

**Adding and Subtracting Radical Expressions**

What is the simplified form of each expression?

A.  $3\sqrt{5x} - 2\sqrt{5x} = (3-2)\sqrt{5x} = \sqrt{5x}$

B.  $6x^2\sqrt{7} + 4x\sqrt{5}$  Radicals are different

C.  $12\sqrt[3]{7xy} - 8\sqrt[5]{7xy}$  index is not the same

D.  $3x\sqrt{xy} + 4x\sqrt{xy} = (3x+4x)\sqrt{xy} = 7x\sqrt{xy}$

**Simplifying Before Adding or Subtracting**

What is the simplest form of the expression?  $\sqrt{12} + \sqrt{75} - \sqrt{3}$

$$\begin{aligned} &\sqrt{12} \\ &\sqrt{4}\sqrt{3} \\ &2\sqrt{3} \end{aligned}$$

$$\begin{aligned} &\sqrt{75} \\ &\sqrt{25}\sqrt{3} \\ &5\sqrt{3} \end{aligned}$$

$$2\sqrt{3} + 5\sqrt{3} - \sqrt{3}$$

$$(2 + 5 - 1)\sqrt{3}$$

$$\boxed{6\sqrt{3}}$$

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Radical expressions that have the same index and radicand

distinctive property

$$(a+b)\sqrt{x} \quad (a-b)\sqrt{x}$$

$$= (3-2)\sqrt{x} = \sqrt{x}$$

Radicals are different

index is not the same

$$(8x+4x)\sqrt{x} = 12x\sqrt{x}$$

$$2\sqrt{2} \quad \sqrt{2}\sqrt{2}$$

$$2\sqrt{3} \quad \sqrt{4}\sqrt{3}$$

$$2\sqrt{3} + 2\sqrt{3} - \sqrt{3}$$

$$(2+2-1)\sqrt{3}$$

$$\boxed{3\sqrt{3}}$$

**Combining Radical Expressions: Products**

When written in radical form, you can multiply two radicals only if the index is the same.

1) Multiply the coefficients

2) Multiply the radicands

3) Simplify!

If  $\sqrt[n]{a}$  and  $\sqrt[n]{b}$  are real numbers, then  $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{a \cdot b}$ .

**Multiplying Radical Expressions**

Can you simplify the product of the radical expressions? Explain.

A.  $\sqrt[3]{6} \cdot \sqrt{2}$

NO the index is different

B.  $\sqrt[3]{-4} \cdot \sqrt[3]{2}$

$$\sqrt[3]{-4 \cdot 2} = \sqrt[3]{-8} = -2$$

**Simplifying a Radical Expression**

What is the simplest form of  $\sqrt[3]{54x^5}$ ?

$\sqrt[3]{54x^5} = \sqrt[3]{3^3 \cdot 2 \cdot x^3 \cdot x^2}$	Find all perfect cube factors
$\sqrt[3]{3^3 x^3} \cdot \sqrt[3]{2x^2}$	$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$
$3x \sqrt[3]{2x^2}$	Simplify.

**Simplifying a Product**

What is the simplest form of  $\sqrt{72x^3y^2} \cdot \sqrt{10xy^3}$ ?

$$\sqrt{72 \cdot 10 \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y \cdot y}$$

$$\sqrt{720 x^4 y^5}$$

$$\sqrt{144 \cdot 5 x^4 y^4 \cdot y} = 12x^2 y^2 \sqrt{5y}$$

radicals  
coefficient

the same  
the index is

$$\sqrt[n]{a \cdot b}$$

no the index is different

$$\sqrt[3]{-4 \cdot 2} = \sqrt[3]{-8} = 2$$

$$\sqrt[3]{24x^2} = \sqrt[3]{3^3 \cdot 2 \cdot x^2}$$

$$3\sqrt[3]{2x^2} \cdot \sqrt[3]{3x^2}$$

$$8x^3 \sqrt[3]{2x^2}$$

$$\sqrt[3]{125 \cdot 10 \cdot xxx \cdot xxx \cdot xxx}$$

$$\sqrt[3]{1250 \cdot xxx}$$

$$\sqrt[3]{15x} = \sqrt[3]{15x^3 \cdot xxx}$$

Combining Radical Expressions: Quotients

If  $\sqrt[n]{a}$  and  $\sqrt[n]{b}$  are real numbers and  $b \neq 0$ , then  $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$ .

Dividing Radical Expressions

What is the simplest form of the quotient?

<p>A. <math>\frac{\sqrt{18x^5}}{\sqrt{2x^3}}</math> <math>\sqrt{\frac{18x^5}{2x^3}}</math></p>	<p>B. <math>\frac{\sqrt[3]{162y^5}}{\sqrt[3]{3y^2}}</math> <math>\sqrt[3]{\frac{162y^5}{3y^2}}</math></p>
<p><math>\frac{18}{2} = 9</math>    <math>5-3 = 2</math></p>	<p><math>\frac{162}{3} = 54</math>    <math>5-2 = 3</math></p>
<p><math>= \sqrt{9x^2}</math></p>	<p><math>\sqrt[3]{54y^3}</math></p>
<p><math>= 3x</math></p>	<p><math>\sqrt[3]{27} \sqrt[3]{2} \sqrt[3]{y^3}</math></p>
	<p><math>3y \sqrt[3]{2}</math></p>

Rationalize the Denominator

Rewrite the expression so that there are no radicals in any denominator and no denominator in any radicals.

$$\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{4}} = \frac{\sqrt{2}}{2}$$

Rationalizing the Denominator

What is the simplest form of  $\sqrt[3]{\frac{5x^2}{12y^2z}}$ ?

<p><math>\sqrt[3]{\frac{5x^2}{2^2 \cdot 3 \cdot y^2 \cdot z}}</math></p>	<p>The radicand in the denominator needs 2, 3, y, and z to make the factors perfect cubes.</p>
<p><math>\sqrt[3]{\frac{5x^2}{2^2 \cdot 3 \cdot y^2 \cdot z}} \cdot \frac{\sqrt[3]{2 \cdot 3^2 \cdot y \cdot z^2}}{\sqrt[3]{2 \cdot 3^2 \cdot y \cdot z^2}}</math></p>	<p>Multiply the numerator and denominator by <math>\sqrt[3]{2 \cdot 3^2 \cdot yz^2}</math>.</p>
<p><math>\frac{\sqrt[3]{90x^2yz^2}}{\sqrt[3]{2^3 \cdot 3^3 \cdot y^3 \cdot z^3}}</math></p>	

$$\sqrt[3]{\frac{27x^3}{18x^2}} = \sqrt[3]{\frac{3x}{2}}$$

$$= \sqrt[3]{\frac{3x}{2}}$$

$$\sqrt[3]{\frac{37x^2}{105x^2}} = \sqrt[3]{\frac{37}{105}}$$

$$= \sqrt[3]{\frac{37}{105}}$$

denominator

radicals

denominator

radicals

$$\frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\sqrt[3]{\frac{27x^3}{105x^2}} = \sqrt[3]{\frac{3x}{105}}$$

$$= \sqrt[3]{\frac{3x}{105}}$$

$\frac{\sqrt[3]{90 \cdot x^2 \cdot y \cdot z^2}}{2 \cdot 3 \cdot y \cdot z}$	
$\frac{\sqrt[3]{90x^2 \cdot y \cdot z^2}}{6yz}$	Simplify.

### Multiplying Binomial Radical Expressions

What is the product of each radical expression?

A. $(4 + 2\sqrt{2})(5 + 4\sqrt{2})$	FOIL
$20 + 16\sqrt{2} + 10\sqrt{2} + (2\sqrt{2} \cdot 4\sqrt{2})$	Distribute
$20 + 26\sqrt{2} + 16$	Multiply
$36 + 26\sqrt{2}$	Combine like terms
B. $(3 - \sqrt{7})(5 + \sqrt{7})$	
$15 + 3\sqrt{7} - 5\sqrt{7} - 7$	Distribute.
$15 - 7 + 3\sqrt{7} - 5\sqrt{7}$	Multiply and combine like radicals
$8 - 2\sqrt{7}$	Simplify.

### Multiplying Conjugates

Conjugates are expressions like  $a + \sqrt{b}$  and  $a - \sqrt{b}$ , they differ only in the signs of the second term.

What is the product of  $(5 - \sqrt{7})(5 + \sqrt{7})$ ?

$$25 + 5\sqrt{7} - 5\sqrt{7} - 7$$

$$(18)$$

$$\frac{\sqrt[3]{10 \cdot x \cdot y \cdot z}}{2 \cdot 3 \cdot y \cdot z} \cdot \frac{\sqrt[3]{10 \cdot x \cdot y \cdot z}}{2 \cdot 3 \cdot y \cdot z}$$

FOIL

$$\begin{aligned} & 30 + 10\sqrt{z} + 10\sqrt{z} + (3yz \cdot yz) \\ & 30 + 20\sqrt{z} + 10 \\ & 30 + 20\sqrt{z} \end{aligned}$$

$$\begin{aligned} & r - \sqrt{a} - \sqrt{a} + 2a \\ & r - 2\sqrt{a} + 2a \\ & r - 2\sqrt{a} \end{aligned}$$

$$\sqrt{a} - a \quad \sqrt{a} + a$$

$$r - \sqrt{a} - \sqrt{a} + 2a$$

(8)



Rationalizing the Denominator

$\frac{3\sqrt{2}}{\sqrt{5}-\sqrt{2}}$	
$\frac{3\sqrt{2}}{\sqrt{5}-\sqrt{2}} \cdot \frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}+\sqrt{2}} =$	Multiply. Use the conjugate of the denominator.
$\frac{3\sqrt{2}(\sqrt{5}+\sqrt{2})}{3}$	The radicals in the denominator cancel out.
$\frac{3\sqrt{10} + 3\sqrt{4}}{3}$	Distribute $\sqrt{2}$ in the numerator.
$\sqrt{10} + 2$	Simplify.
$2 + \sqrt{10}$	

$$(\sqrt{5}-\sqrt{2})(\sqrt{5}+\sqrt{2})$$

$$\sqrt{25} + \sqrt{10} - \sqrt{10} - \sqrt{4}$$

$$5 - 2 = 3$$

Do you know HOW?

Multiply, if possible. Then simplify.

- $\sqrt{2} \cdot \sqrt{5} = \sqrt{10}$
- $\sqrt[3]{-27} \cdot \sqrt[3]{4} = -3 \cdot \sqrt[3]{4}$
- $\sqrt[3]{2} \cdot \sqrt[2]{7} \rightarrow$  NOT like index
- $\sqrt{3} \cdot \sqrt{-4} = \sqrt{-4 \cdot 3} = \sqrt{-12} =$

Divide and simplify.

$$5. \frac{\sqrt[3]{15x^2}}{\sqrt[3]{5x}} \cdot \frac{\sqrt[3]{5^2 \cdot x^2}}{\sqrt[3]{5^2 \cdot x^2}} = \frac{\sqrt[3]{375 \cdot x^4}}{5x} = \frac{\sqrt[3]{5 \cdot 5 \cdot 5 \cdot 3 \cdot x \cdot x \cdot x \cdot x}}{5x}$$

$$6. \frac{\sqrt{21x^{10}}}{\sqrt{7x^5}} = \frac{5x \sqrt[3]{3 \cdot x}}{5x} = \sqrt[3]{3x}$$

b)  $\frac{\sqrt{21x^{10}}}{\sqrt{7x^5}} \cdot \frac{\sqrt{7x}}{\sqrt{7x}} = \frac{\sqrt{147x^{11}}}{\sqrt{49x^6}} = \frac{\sqrt{7 \cdot 7 \cdot 3 \cdot x^2 \cdot x^2 \cdot x^2 \cdot x^2 \cdot x^2 \cdot x}}{7x^3} = \frac{7x^5 \sqrt{3x}}{7x^3} = x^2 \sqrt{3x}$

$$= \frac{\frac{3\sqrt{2}}{\sqrt{10-13}} \cdot \frac{2\sqrt{2}}{\sqrt{10+13}}}{\frac{3\sqrt{2}(\sqrt{2}+\sqrt{13})}{3}} = \frac{10 + \sqrt{2}}{3}$$

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$$(10 - \sqrt{2})(10 + \sqrt{2}) = 100 - 2 = 98$$

$$= 10$$

$$= 3\sqrt{2}$$

→ not like in ex

$$= \sqrt{2} \cdot 3 = \sqrt{18} = 3\sqrt{2}$$

$$\frac{\sqrt{2} \cdot \sqrt{2} \cdot \sqrt{2} \cdot \sqrt{2} \cdot \sqrt{2} \cdot \sqrt{2}}{2x} = \frac{\sqrt{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}}{2x} = \frac{2^3 \cdot 2}{2x} = \frac{2^4}{2x} = \frac{16}{2x} = \frac{8}{x}$$

$$\frac{2x \sqrt{3 \cdot x}}{2x} = \sqrt{3x}$$

**Do you know HOW?**

Simplify if possible.

1.  $10\sqrt{6} + 2\sqrt{6}$

2.  $3\sqrt{2} + 4\sqrt[3]{2}$

3.  $8\sqrt{3x} - 5\sqrt{3x}$

4.  $5\sqrt{3} + \sqrt{12}$

Multiply.

5.  $(4 + \sqrt{3})(4 - \sqrt{3})$

6.  $(5 + 2\sqrt{5})(7 + 4\sqrt{5})$

7.  $(2 + 3\sqrt{2})(1 - 3\sqrt{2})$

①  $12\sqrt{6}$

② Not like  
Radicals

③  $3\sqrt{3}x$

④ Not like  
Radicals

⑤  $16 - 4\sqrt{3} + 4\sqrt{3} - \sqrt{9}$

$16 - 3 = 13$

⑥  $35 + 20\sqrt{5} + 14\sqrt{5} + 8\sqrt{25}$

$75 + 34\sqrt{5}$

⑦  $2 - 6\sqrt{2} + 3\sqrt{2} - 9\sqrt{4}$

$2 - 3\sqrt{2} - 18$

$-16 - 3\sqrt{2}$



# NOTES

A radical equation is an equation that has a variable in a radicand or a variable with a rational exponent.

## Solving a Square Root Equation

What is the solution of  $3 + \sqrt{2x-3} = 8$ ?

$$\begin{aligned} \sqrt{2x-3} &= 5 \\ (\sqrt{2x-3})^2 &= (5)^2 \\ 2x-3 &= 25 \\ 2x &= 28 \\ x &= 14 \end{aligned}$$

Isolate the radical expression.

Square each side.

Add 3 to each side.

Divide each side by 2.

What is the solution of  $\sqrt{4x+1} - 5 = 0$ ?

$$\begin{aligned} \sqrt{4x+1} &= +5 \\ (\sqrt{4x+1})^2 &= (+5)^2 \\ 4x+1 &= 25 \\ 4x &= 24 \\ x &= 6 \end{aligned}$$

Isolate the radical expression.

Square each side.

Subtract 1 from each side.

Divide each side by 4.

## Solving Other Radical Equations

To solve equations of the form  $x^{\frac{m}{n}} = k$ , raise each side of the equation to the power  $\frac{n}{m}$ , the reciprocal of  $\frac{m}{n}$ . If either m or n is even, then

$$\left(x^{\frac{m}{n}}\right)^{\frac{n}{m}} = |x|.$$

Example 1 - What is the solution of  $3(x+1)^{\frac{2}{3}} = 12$ ?

$$(x+1)^{\frac{2}{3}} = 4$$

$$\left[(x+1)^{\frac{2}{3}}\right]^{\frac{3}{2}} = (4)^{\frac{3}{2}}$$

$$x+1 = 8$$

$$|x+1| = 8$$

$$x+1 = 8 \text{ OR } x+1 = -8$$

$$x = 7 \text{ OR } x = -9$$

Divide each side by 3.

Raise each side to the  $\frac{3}{2}$  power.

Since the numerator of  $\frac{2}{3}$  is even,  $\left(x^{\frac{2}{3}}\right)^{\frac{3}{2}} = |x|$ .

Example 2 - What is the solution of  $3\sqrt[5]{(x+1)^3} + 1 = 25$ ?

$$3(x+1)^{\frac{3}{5}} + 1 = 25$$

$$3(x+1)^{\frac{3}{5}} = 24$$

$$(x+1)^{\frac{3}{5}} = 8$$

$$\left[(x+1)^{\frac{3}{5}}\right]^{\frac{5}{3}} = 8^{\frac{5}{3}}$$

$$x+1 = 32$$

$$x = 31$$

Rewrite the radical using a rational exponent.

Subtract 1 from each side.

Divide each side by 3.

Raise each side to the  $\frac{5}{3}$  power.

Simplify.

Subtract 1 from each side.

### Extraneous Solutions

When you raise each side of an equation to a power, it is possible to introduce extraneous solutions.

You must check solutions when solving radical equations.

*A correct solution will give a true statement.*

*An extraneous solution will give a false statement.*

Checking for Extraneous Solutions

What is the solution of  $\sqrt{x+7} - 5 = x$ ? Check your results.

- $\sqrt{x+7} = x+5$  Isolate the radical.
- $(\sqrt{x+7})^2 = (x+5)^2$  Square each side.
- $x+7 = x^2+10x+25$  Simplify.
- $x^2+9x+18=0$  Combine like terms.
- $(x+3)(x+6)$  Factor.
- $x=-3 \quad x=-6$  Zero-Product Property

$\sqrt{x+1} = \sqrt{1+x}$   
 $(\sqrt{x+1})^2 = (\sqrt{1+x})^2$   
 $1+x+1 = 1+x+1$   
 $2 = 2$   
 $x = x$   
 $(1) = (1)$   
 $x = x$

Check

$\sqrt{x+7} - 5 = x$

$\sqrt{x+7} - 5 = x$

$\sqrt{-3+7} - 5 = -3$

$\sqrt{-6+7} - 5 = -6$

$\sqrt{4} - 5 = -3$

$\sqrt{1} - 5 = -6$

$2 - 5 = -3$

$1 - 5 = -6$

$-3 = -3 \checkmark$

$-4 \neq -6$

$x = -6$  extraneous solution

Stop Here

Solving an Equation with Two Radicals

If an equation contains two radical expressions, isolate one of the radicals, then eliminate it.

Isolate the more complicated radical expression first.

In the resulting equation, simplify the expressions before your eliminate the second radical.

Both solutions are correct.  
 $x = 0 \quad x = 4$

Example 1 - What is the solution of  $\sqrt{2x+1} - \sqrt{x} = 1$ ?

$$\sqrt{2x+1} = 1 + \sqrt{x}$$

$$(\sqrt{2x+1})^2 = (1 + \sqrt{x})^2$$

$$2x+1 = x+2\sqrt{x}+1$$

$$x = 2\sqrt{x}$$

$$(x)^2 = (2\sqrt{x})^2$$

$$x^2 = 4x$$

$$x^2 - 4x = 0$$

$$x(x-4)$$

$$x=0 \quad x=4$$

Check your solutions!!

$$\sqrt{2(0)+1} - \sqrt{0} = 1$$

$$\sqrt{1} - 0 = 1$$

$$1 - 0 = 1$$

$$1 = 1 \checkmark$$

$$\sqrt{2(4)+1} - \sqrt{4} = 1$$

$$\sqrt{9} - \sqrt{4} = 1$$

$$3 - 2 = 1$$

$$1 = 1$$

Both solutions are correct.

$$x=0 \text{ and } x=4$$

Isolate the more complicated radical.

Square each side.

Isolate  $2\sqrt{x}$ .

Square each side.

Subtract  $4x$  from each side.

Factor.

Zero-Product Property.



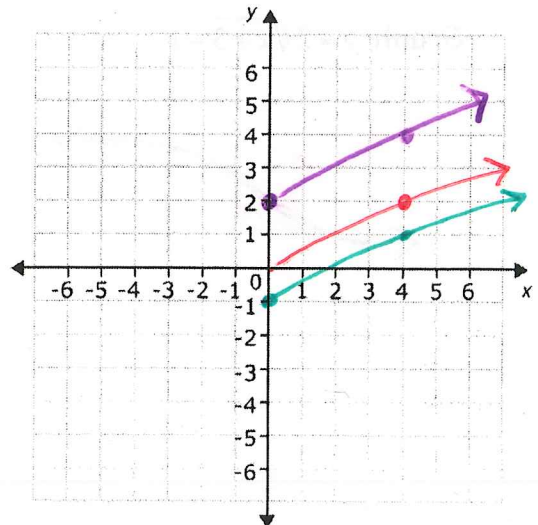
Inverses of functions of the form  $y = x^n$  (with domain restricted as needed) form parent functions  $y = \sqrt[n]{x}$  of families of radicals functions. In particular,  $f(x) = \sqrt{x}$  is the parent for the family of square root functions. Members of this family have the form  $a\sqrt{x-h} + k$ .

Families of Radical Functions Including the Square Root Function		
Parent Function:	Square Root $\sqrt{x}$	Radical $\sqrt[n]{x}$
Reflection in x-axis	$y = -\sqrt{x}$	$y = -\sqrt[n]{x}$
Stretch ( $a > 1$ ), shrink ( $0 < a < 1$ ) by factor $a$ :	$y = a\sqrt{x}$	$y = a\sqrt[n]{x}$
Reflection in x-axis	$y = -a\sqrt{x}$	$y = -a\sqrt[n]{x}$
Translation:		
Horizontal by $h$ :	$y = \sqrt{x-h}$	$y = \sqrt[n]{x-h}$
Vertical by $k$ :	$y = \sqrt{x} + k$	$y = \sqrt[n]{x} + k$
Combined:	$y = \sqrt{x-h} + k$	$y = \sqrt[n]{x-h} + k$

Translating Square Root Functions Vertically

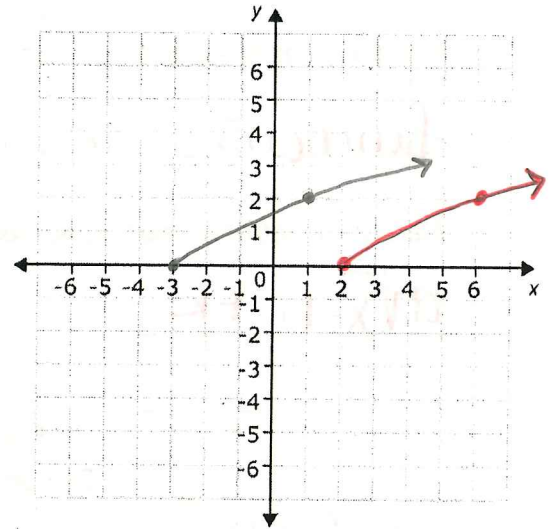
Graph  $y = \sqrt{x} + 2$  and  $y = \sqrt{x} - 1$

$y = \sqrt{x} \rightarrow$  parent function



**Translating Square Root Functions Horizontally**

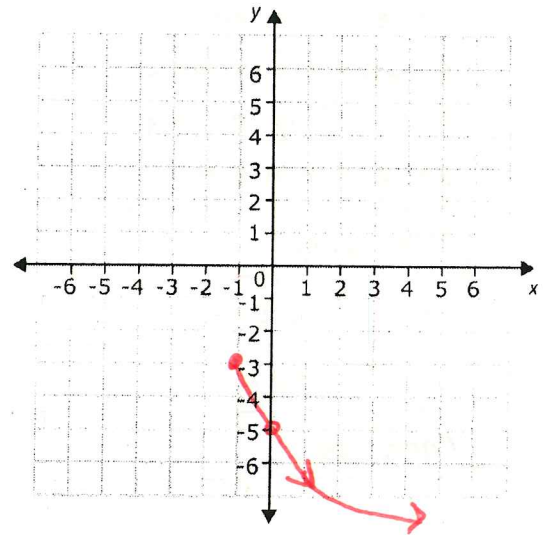
Graph  $y = \sqrt{x+3}$  and  $y = \sqrt{x-2}$



**Graphing Square Root Functions**

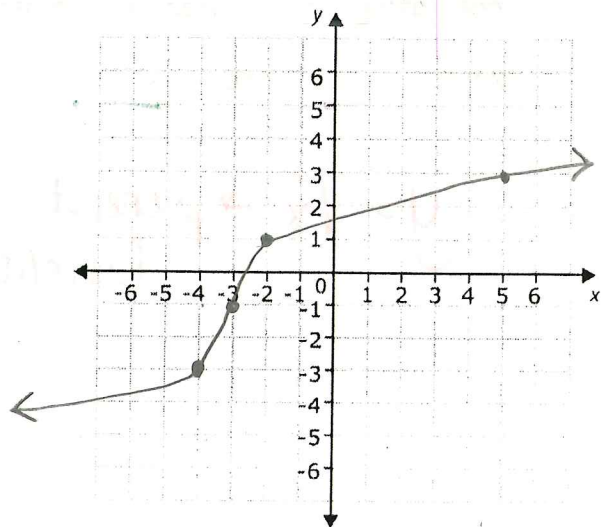
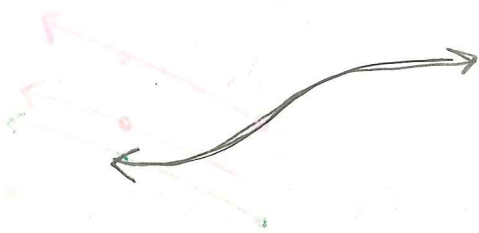
Graph  $y = -2\sqrt{x+1} - 3$

x	$-2\sqrt{x+1} - 3$	y
-1	0	-3
3	$-2\sqrt{4} - 3$	-7
0	$-2\sqrt{1} - 3$	-5



**Graphing Cube Root Functions**

Graph  $y = 2\sqrt[3]{x+3} - 1$



### Solving Square Root Equations by Graphing

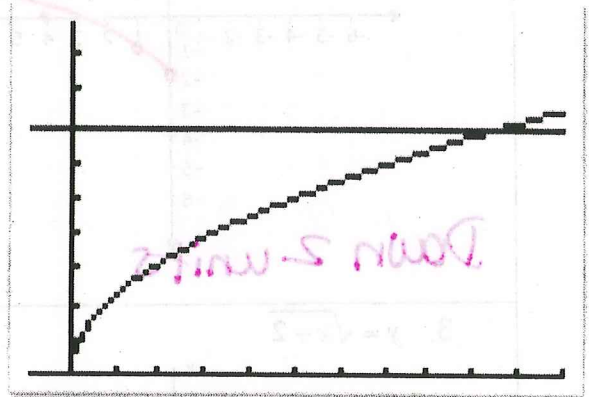
You can model the time  $t$ , in seconds; an object takes to reach the ground falling from height  $H$ , in meters, by  $t(H) = \sqrt{\frac{2H}{g}}$ . The value of  $g$  is  $9.81 \text{ m/s}^2$ . From what height does an object fall if it takes 7 seconds to reach the ground?

For  $t = 7$ , solve the equation  $7 = \sqrt{\frac{2H}{g}}$

1. Graph  $Y_1 = \sqrt{2X/9.81}$ ,  $Y_2 = 7$
2. Adjust Window as follows:

```

WINDOW
Xmin=-25
Xmax=275
Xscl=25
Ymin=0
Ymax=10
Yscl=1
Xres=1
  
```

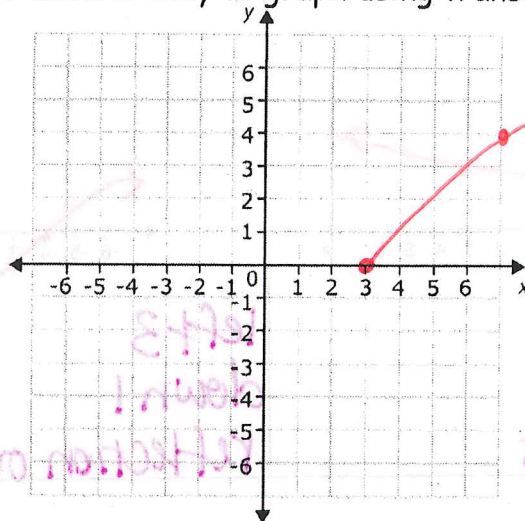


3. Use Intersect function to find intersection of two functions.

4. The answer is 240 m.

### Rewriting Radical Functions

Rewrite  $y = \sqrt{4x-12}$  to make it easy to graph using transformations. Describe the graph.



$$y = \sqrt{4(x-3)}$$

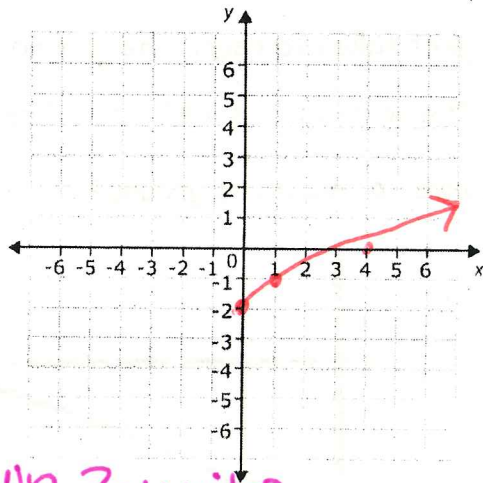
$$y = 2\sqrt{x-3}$$

Right by 3

Stretch by 2

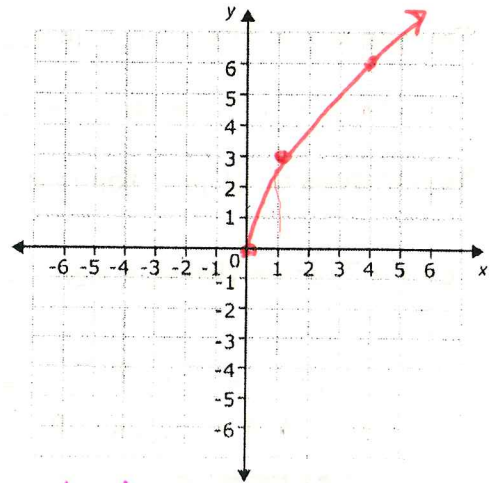
2nd trace #5

1.  $y = \sqrt{x} - 2$



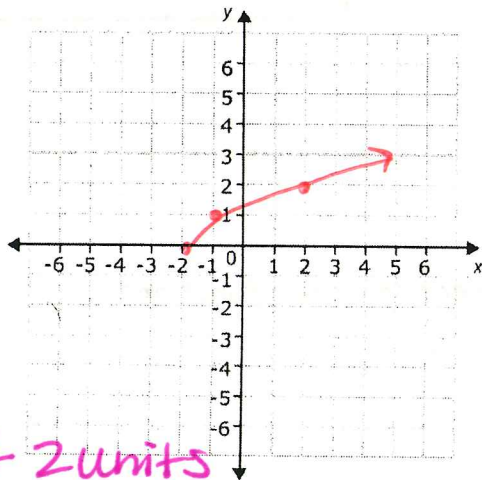
Down 2 units

2.  $y = 3\sqrt{x}$



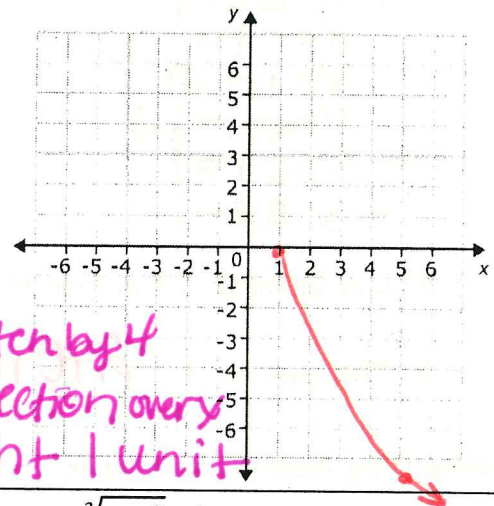
Stretch by 3

3.  $y = \sqrt{x+2}$



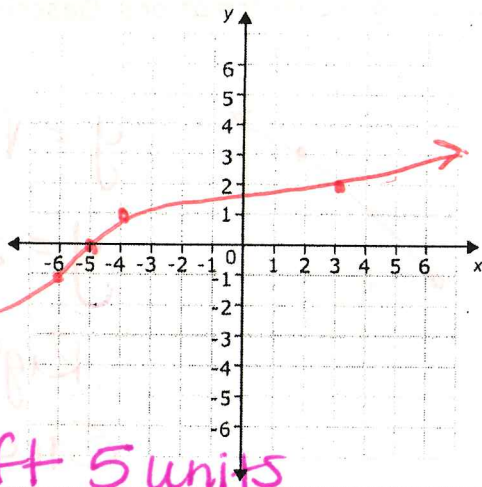
left 2 units

4.  $y = -4\sqrt{x-1}$



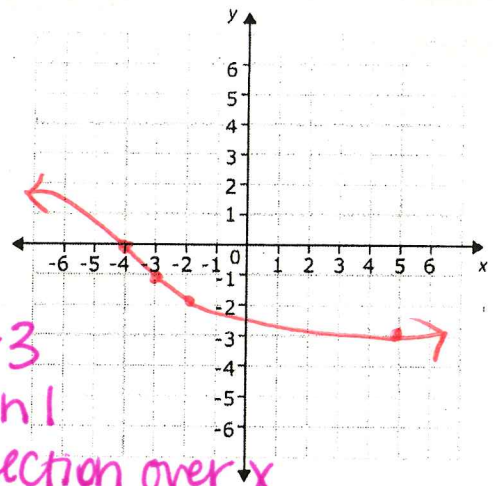
stretch by 4  
reflection over  
right 1 unit

5.  $y = \sqrt[3]{x+5}$



left 5 units

6.  $y = -\sqrt[3]{x+3} - 1$



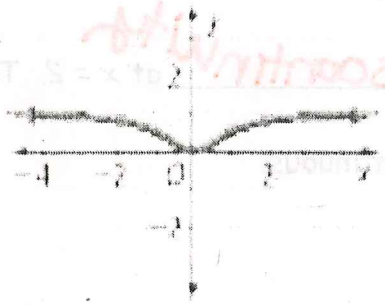
left 3  
down 1  
reflection over x

A rational function is a function that you can write in the form  $f(x) = \frac{P(x)}{Q(x)}$  where  $P(x)$  and  $Q(x)$  are polynomial functions.

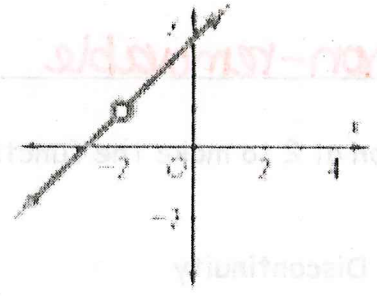
The domain of  $f(x)$  is all real numbers except those values for which  $Q(x) = 0$ .

Graphs of three rational functions:

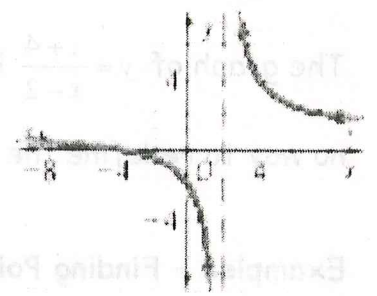
$$y = \frac{x^2}{x^2 + 1}$$



$$y = \frac{(x+3)(x+2)}{(x+2)}$$



$$y = \frac{(x+4)}{(x-2)}$$



For the first rational function,  $y = \frac{x^2}{x^2 + 1}$ , there is no value of  $x$  that makes the denominator 0.

The graph is a continuous graph because it has not jumps, breaks, or holes.

For the second rational function,  $y = \frac{(x+3)(x+2)}{(x+2)}$ ,  $x$  cannot be -2. For the third rational

function,  $y = \frac{x+4}{x-2}$ ,  $x$  cannot be 2.

The second and third graphs are discontinuous graphs.

①

②

**Point of Discontinuity**

If  $a$  is a real number for which the denominator of a rational function  $f(x)$  is zero, then  $a$  is not in the domain of  $f(x)$ .

The graph of  $f(x)$  is not continuous at  $x = a$  and the function has a point of discontinuity at  $x = a$ .

The graph of  $y = \frac{(x+3)(x+2)}{(x+2)}$  has a removable discontinuity at  $x = -2$ . The hole in the graph is called a removable discontinuity because you could make the function continuous by redefining it at  $x = -2$  so that  $f(-2) = 1$ .

The graph of  $y = \frac{x+4}{x-2}$  has a non-removable discontinuity at  $x = 2$ . There is no way to redefine the function at 2 to make the function continuous.

**Examples - Finding Points of Discontinuity**

What are the domain and points of discontinuity of each rational function?

Are the points of discontinuity removable or non-removable?

What are the x- and y-intercepts?

A.  $y = \frac{x+3}{x^2-4x+3}$       $\frac{x+3}{(x-3)(x-1)}$      • Domain: all real except  $x=3$  &  $x=1$

• Non-removable points of discontinuity because nothing in the numerator & denominator cancel.

• X-intercept: where the numerator equals 0.  
 $x+3=0$       $x=-3$ : x-int

• y-intercept: let  $x=0 \rightarrow \frac{0+3}{(0-3)(0-1)} = \frac{3}{3} = 1$       $y=1$

$$B. y = \frac{x-5}{x^2+1}$$

Domain all real numbers

No discontinuities

$$x\text{-int: } x = 5$$

$$y\text{-int: } y = -5$$

$$C. y = \frac{x^2 - 3x - 4}{x - 4} = \frac{(x-4)(x+1)}{(x-4)}$$

Domain: all real #'s  
except  $x = 4$

Removable discontinuity @  $x = 4$

$$x\text{-int: } x = -1$$

$$y\text{-int: } y = 1$$

$$\frac{(x-4)(x+1)}{(x-4)} = x+1 = 0 \quad x = -1$$

### Vertical Asymptotes of Rational Functions

The graph of the rational function  $f(x) = \frac{P(x)}{Q(x)}$  has a vertical asymptote at each real zero of

$Q(x)$  if  $P(x)$  and  $Q(x)$  have no common zeros. If  $P(x)$  and  $Q(x)$  have  $(x-a)^m$  and  $(x-a)^n$  as factors, respectively and  $m < n$ , then  $f(x)$  also has a vertical asymptote at  $x = a$ .

### Example - Finding Vertical Asymptotes

What are the vertical asymptotes for the graph of  $y = \frac{(x+1)}{(x-2)(x-3)}$ ?

Since 2 & 3 are zeros & don't cancel w/ num.

$x = 2$  &  $x = 3$  are vertical asymptotes

### Horizontal Asymptote of a Rational Function

To find the horizontal asymptote of the graph of a rational function, compare degree of the numerator (m) to the degree of the denominator (n)

- If  $m < n$ , the graph has horizontal asymptote  $y = 0$  (the  $x$ -axis).
- If  $m > n$ , the graph has no horizontal asymptote.
- If  $m = n$ , the graph has horizontal asymptote  $y = \frac{a}{b}$  where  $a$  is the coefficient of the term of greatest degree in the numerator and  $b$  is the coefficient of the term of greatest degree in the denominator.

### Finding Horizontal Asymptotes

What is the horizontal asymptote for the rational function?

1.  $y = \frac{2x}{x-3}$

$y = \frac{2}{1}$  or  $y = 2$

Degrees are the same

2.  $y = \frac{x-2}{x^2-2x-3}$

$y = 0$       $m < n$

3.  $y = \frac{x^2}{2x-5}$

No horizontal  
Asymptote

$m > n$



Graphing the Rational Function  $y = \frac{x^2 + x - 12}{x^2 - 4}$ .

1. Identify the Horizontal Asymptote:

$$m = n \quad y = \frac{1}{1} = 1$$

2. Identify the Vertical Asymptotes:

$$\frac{(x+4)(x-3)}{(x+2)(x-2)} \rightarrow x=2, x=-2$$

3. Find the x- and y-intercepts:

$$x\text{-int.} \begin{cases} x+4=0 & x=-4 \\ x-3=0 & x=3 \end{cases}$$

$$y\text{-int.} \begin{cases} \frac{(0+4)(0-3)}{(0+2)(0-2)} = 3 \\ y=3 \end{cases}$$

4. Identify Additional Points:

$$\left(-3, -\frac{6}{5}\right) \quad (-1, 4) \quad \left(1, \frac{10}{3}\right) \quad \left(4, \frac{2}{3}\right)$$

