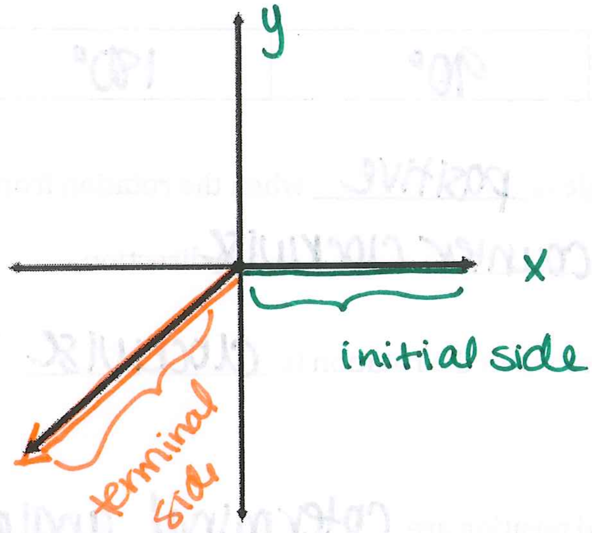
Math III
Notes 5-1 The Unit Circle

Name Key
Date ________ Period ________

Working With Angles in Standard Position

An angle is in Standard position when the vertex is at the origin and one ray is on the positive x-axis. The ray on the x-axis in the initial side of the angle; the other ray is the terminal side of the angle.

Standard Position

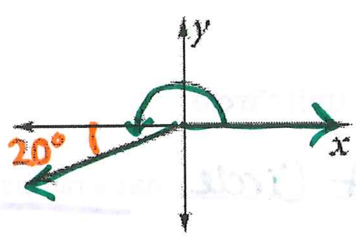


To measure an angle in the standard position, find the amount of rotation from the initial side to the terminal side.

Measuring an Angle in Standard Position

Find the measure of the angle at the right.

The angle measures 20° more than a straight angle of 180°.



Since 180 + 20 = 200°, the measure of the angle is 200°.

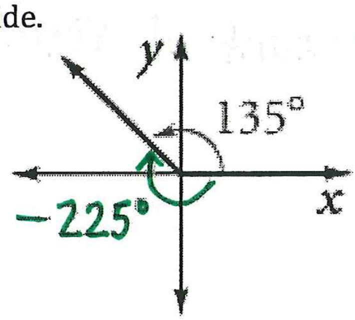
360°	90°	180°	270°

The measure of an angle is positive when the rotation from the initial side to the terminal side is in the counter clockwise direction.

The measure is negative when the rotation is clockwise.

Coterminal Angles

Two angles in standard position are coterminal angles if they have the same terminal side.



Angles that have measures 135° and −225° are coterminal.

Using the Unit Circle

The Unit Circle has a radius of 1 unit and its center at the origin of the coordinate plane.

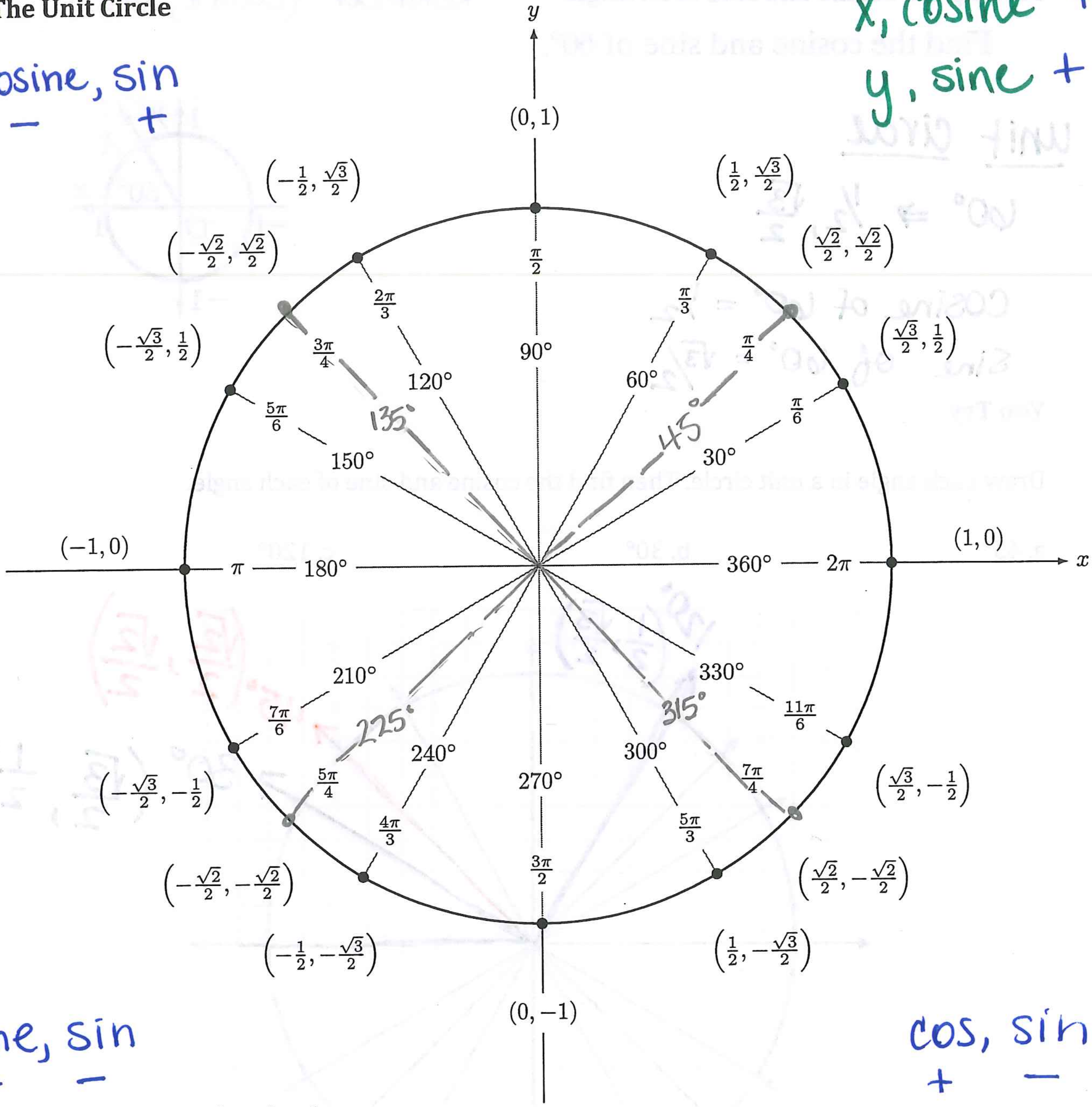
Points on the unit circle are related to periodic functions.

$\frac{\sqrt{3}}{2} = .86602\ldots$

The Unit Circle

cosine, sin
− +

x, cosine +
y, sine +



cosine, sin
− −

cos, sin
+ −

Cosine and Sine of an Angle

Suppose an angle in standard position has measure θ. θ means theta

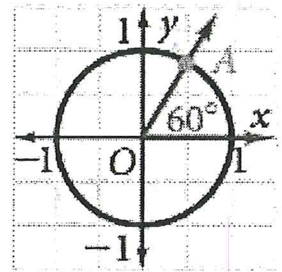
The Cosine of θ (cos θ) is the x-coordinate of the point at which the terminal side of the angle intersects the unit circle.

The sine of θ (sin θ) is the y-coordinate.

(x , y)

Finding the Cosine and Sine of an Angle Remember (cosine, sine)

Find the cosine and sine of 60°.



unit circle

$60° \Rightarrow \frac{1}{2}, \frac{\sqrt{3}}{2}$

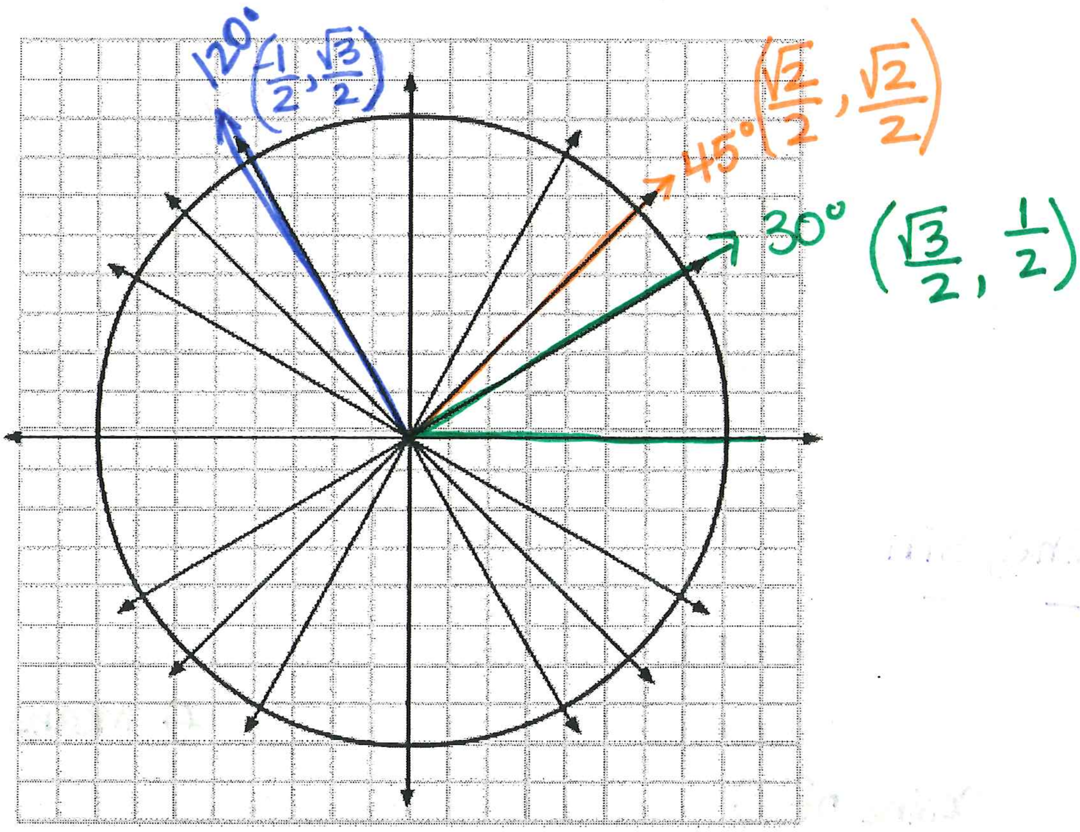
cosine of 60° = $\frac{1}{2}$

sine of 60° = $\frac{\sqrt{3}}{2}$

You Try

Draw each angle in a unit circle. Then find the cosine and sine of each angle.

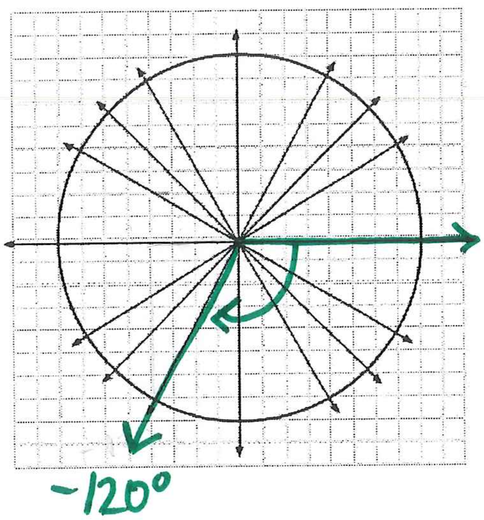
a. 45° b. 30° c. 120°



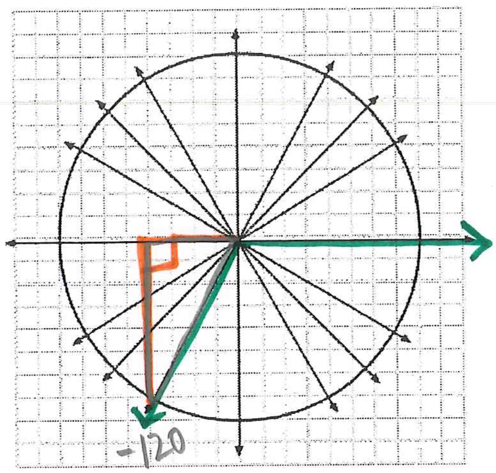
Finding Exact Values of Cosine and Sine

Find the exact values of cos (−120°) and sin (−120°).

Step 1 Sketch an angle of −120°.

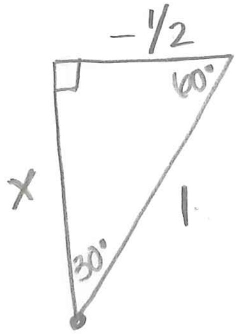


Step 2 Sketch a right triangle.



Step 3 Find the length of each side of the triangle.

$180° - 120° = 60°$



$\left(-1/2, -\frac{\sqrt{3}}{2}\right)$

$a^2 + b^2 = c^2$

$(-1/2)^2 + b^2 = 1^2$

$b^2 = 1 - .25$

$b^2 = .75$

$b = \sqrt{.75} = .866025...$

$\frac{\sqrt{3}}{2} = .866025...$

Math III
Notes 5-2 Radian Measure

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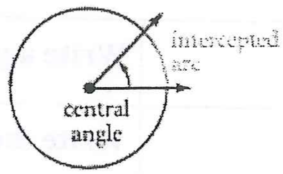
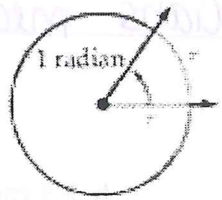
Using Radian Measure



A Central angle of a circle is an angle with a vertex at the center of a circle.

An intercepted arc is the portion of the circle with endpoints on the sides of the central angle and remaining points within the interior of the angle.



When a central angle intercepts an arc that has the same length as a radius of the circle, the measure of the angle is defined to be one radian.

Because the circumference of a circle is $2\pi r$, there are 2π radians in any circle.

Since 2π radians $= 360°$, and therefore π radians $= 180°$, you can use a proportion such as $\dfrac{d°}{180°} = \dfrac{r \text{ radians}}{\pi \text{ radians}}$ to convert between degrees and radians.

Using a Proportion to Convert Degrees to Radians

a. Find the radian measure of an angle of 60°.

$\dfrac{60}{180} = \dfrac{r}{\pi}$	**Write a proportion.**
$60\pi = 180 \cdot r$	**Write the cross-products.**
$r = \dfrac{60\pi}{180}$	**Divide each side by 180.**
$r = \pi/3 \approx 1.05$	**Simplify.**
An angle of 60° measures $\pi/3$ or 1.05 radians	

$C = \pi/d$ or $\pi = \dfrac{C}{d}$

b. Find the degree measure of $\frac{5\pi}{2}$ radians.

$\frac{5\pi}{2} / \pi = \frac{d}{180}$	**Write a proportion.**
$\frac{5\pi}{2}(180) = \pi(d)$	**Write the cross-products.**
$\frac{5 \cdot 180 \cdot \cancel{\pi}}{2\cancel{\pi}} = d$	**Divide each side by π.**
$450^\circ = d$	**Simplify.**
An angle of $5\pi/2$ radians measures 450°	

You Try:

a. 85° to radians.

$\frac{85}{180} = \frac{r}{\pi}$

$85\pi = r \cdot 180$

$\frac{85\pi}{180} = r$

$r = 1.48$

b. 2.5 radians to degrees.

$\frac{d}{180} = \frac{2.5}{\pi}$

$d\pi = 2.5(180)$

$d = 143.24^\circ$

Converting Between Radians and Degrees

To convert degrees to radians multiply by $\frac{\pi \text{ radians}}{180^\circ}$.

To convert radians to degrees multiply by $\frac{180^\circ}{\pi \text{ radians}}$.

Example

Find the degree measure of an angle of $-\frac{3\pi}{4}$ radians.

$-\frac{3\pi}{4} \cdot \frac{180}{\pi} = \frac{-3 \cdot 180}{4} = -135^\circ$

You can find the Sine and Cosine of angles in radian measure by using the calculator in radian mode or by using your unit circle.

Finding Cosine and Sine of Radian Measures

Find the exact values of cos ($\frac{\pi}{4}$ radians) and sin ($\frac{\pi}{4}$ radians) using the unit circle and approximate values using the calculator.

$\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

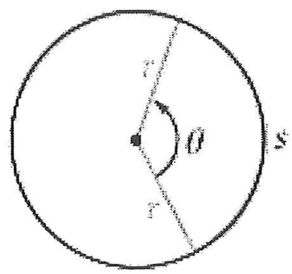
$\frac{\sqrt{2}}{2} = .707106$

$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

Finding the Length of an Arc

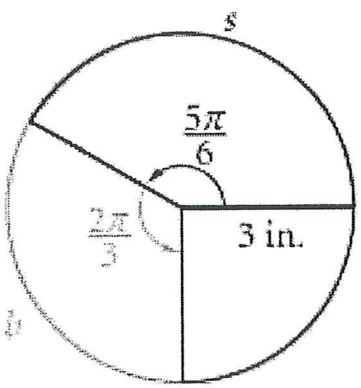
For a circle of radius *r* and a central angle of measure Θ (in radians), the length *s* of the intercepted arc is:

$s = r\theta$



Example

Use the circle below. Find the length of *s* to the nearest tenth.



$s = (3)(5\pi/6)$

$s = \frac{5 \cdot 3 \cdot \pi}{6} = 7.9$

$b = \frac{2\pi}{3} \cdot 3 = \frac{2 \cdot 3 \cdot \pi}{3} = 6.3$

Name Key Class Date

Key

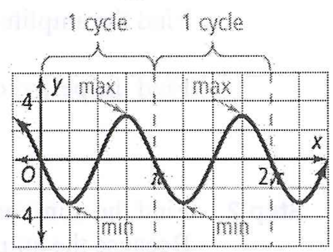
5-3 Notes
The Sine Function

$y = a\sin(bx + c) + d$

A sine curve is the graph of a sine function. You can identify a sine curve by its *amplitude* and *period*. Amplitude is one-half the vertical distance between the maximum and minimum values. The period is the horizontal length of one cycle.

Problem

Use the graph of $y = -3 \sin 2x$, where x is measured in radians, at the right. What are the amplitude and period of the sine curve?



Amplitude

The maximum value of the sine curve is 3.

The minimum value of the sine curve is –3.

One-half the difference of these values is $\frac{(3-(-3))}{2} = \frac{6}{2} = 3$.

The amplitude of the curve is 3.

> The **amplitude** equals the absolute value of the coefficient of the function.

Period

Between 0 and 2π, the graph cycles 2 times.

To get the length of one cycle, divide 2π by the number of cycles between 0 and 2π.

The period of the curve is $\frac{2\pi}{2} = \pi$.

> The **number of cycles** between 0 and 2π equals the coefficient of x in the function.

Summary

For all sine functions written in the form $y = a \sin b\theta$, where $a \neq 0$, $b > 0$, and θ is measured in radians:

amplitude $= |a|$ period $= \frac{2\pi}{b}$

Exercises

Find the amplitude and period of each sine function.

1. $y = \frac{1}{2}\sin 3\theta$

amp: $\frac{1}{2}$ P: $\frac{2\pi}{3}$

2. $y = \sin 5\theta$

A: 1 P: $\frac{2\pi}{5}$

3. $y = 4\sin\frac{4}{3}\pi\theta$

A: 4 $P = \frac{2\pi}{4/3} = \frac{3\pi}{2}$

4. $y = \frac{3}{2}\sin\theta$

A: $\frac{3}{2}$ P: $\frac{2\pi}{1}$

5. $y = -2\sin\frac{3}{4}\theta$

A: 2 $P = \frac{2\pi}{(3/4)} = \frac{8\pi}{3}$

6. $y = \pi \sin 2\theta$

A: π

P: $\frac{2\pi}{2} = \pi$

Name Class Date

5-3 Notes (continued)

The Sine Function

* graph paper for sine function

Problem

What is the graph of two cycles of $y = 2\sin\frac{1}{2}\theta$?

Step 1 Compare the function to $y = a \sin b\theta$. $a = 2$ and $b = \frac{1}{2}$

Find the amplitude. $|a| = |2| = 2$

Find the period of the curve. $\frac{2\pi}{b} = \frac{2\pi}{\frac{1}{2}} = 4\pi s$

Step 2 Find the minimum and maximum of the curve.
Because the amplitude is 2, the maximum is 2 and the minimum is –2.

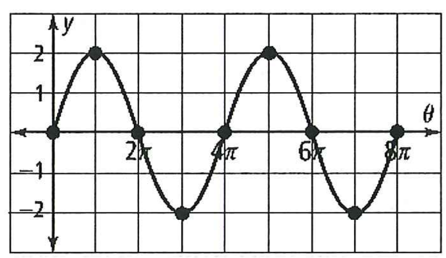
Step 3 Make a table of values. Choose θ-values at intervals of one-fourth the period: $\frac{4\pi}{4} = \pi$.

The y-values cycle through the pattern *zero-max-zero-min-zero.*

θ	0	π	2π	3π	4π	5π	6π	7π	8π
y	0	2	0	–2	0	2	0	–2	0

Step 4 Plot the points from the table.

Step 5 Draw a smooth curve through the points.



Exercises

Graph each function.

7. $y = 2\sin 2\theta$

8. $y = \sin\frac{1}{3}\theta$

9. $y = \frac{1}{2}\sin\theta$

10. $y = -2\sin\frac{1}{2}\theta$

11. $y = -\sin 3\theta$

12. $y = -\frac{1}{4}\sin\theta$

5-3 Notes

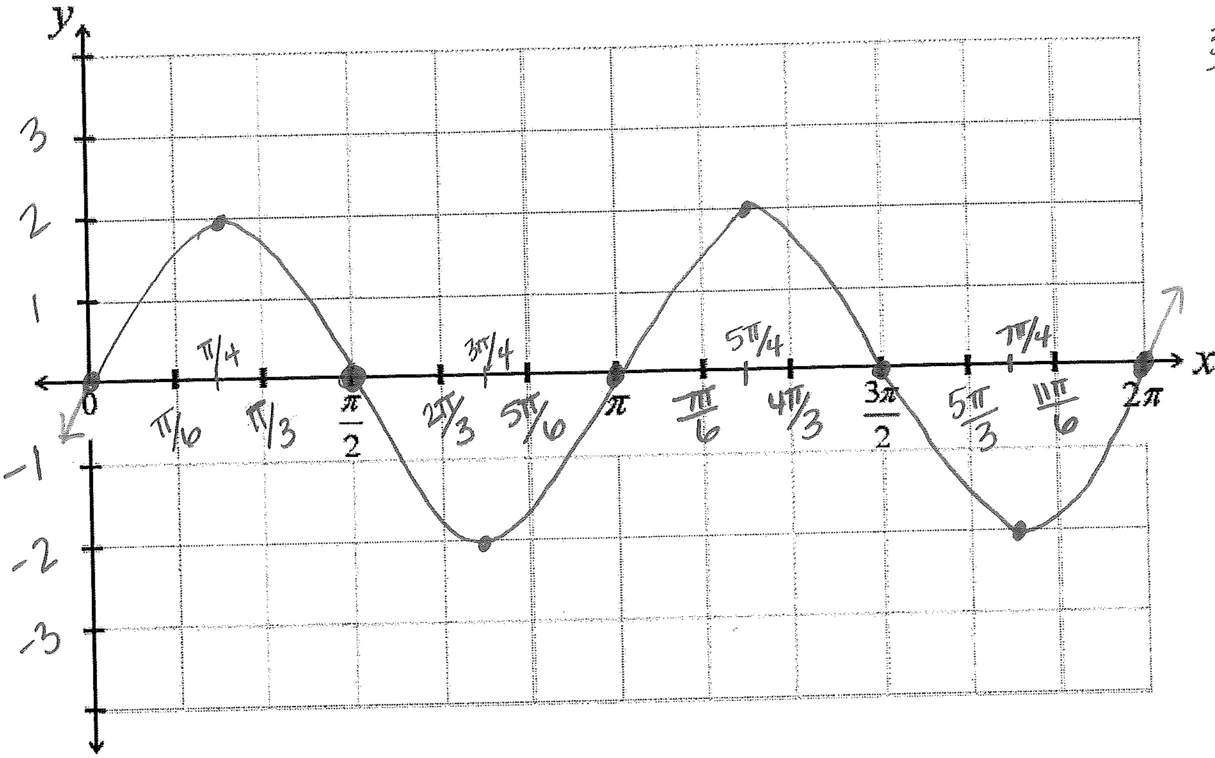
⑦ $y = 2\sin 2\theta$

$\frac{\pi}{4}$

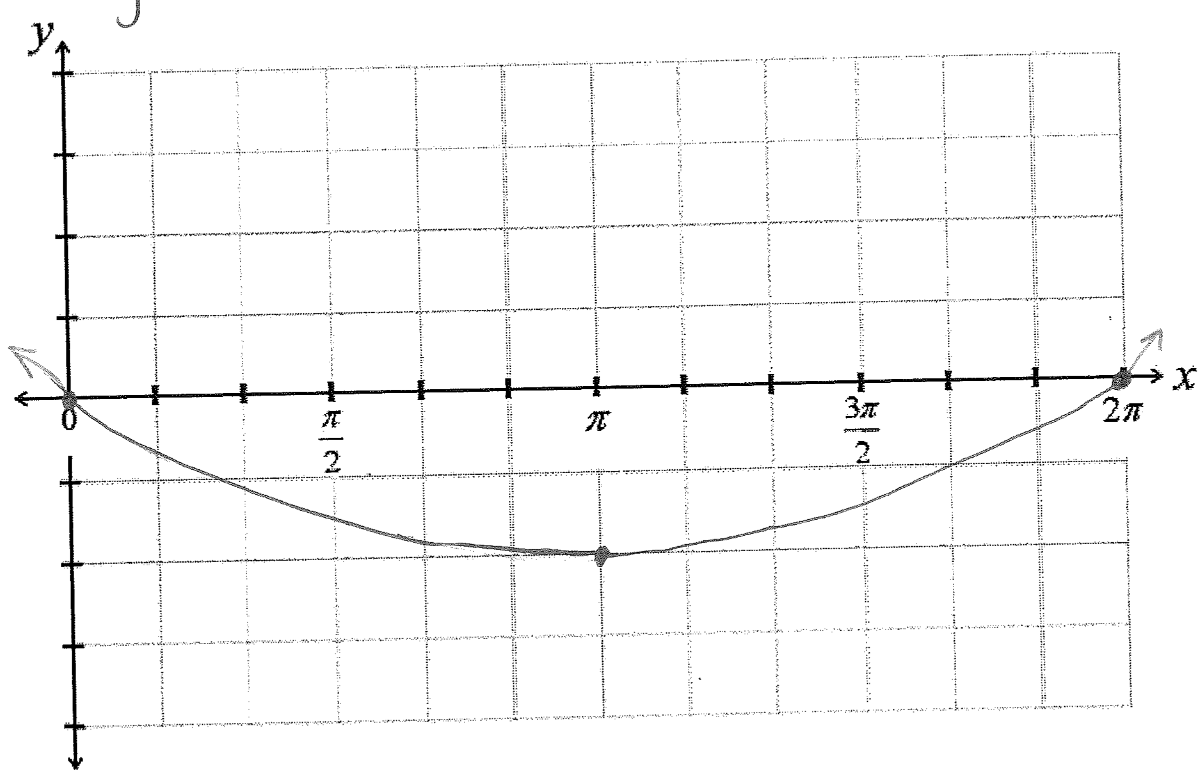
$\frac{\pi}{2} \cdot \frac{1}{3} = \frac{\pi}{6}$

$\frac{\pi}{6} + \frac{\pi}{6} = \frac{2\pi}{6} = \frac{\pi}{3}$

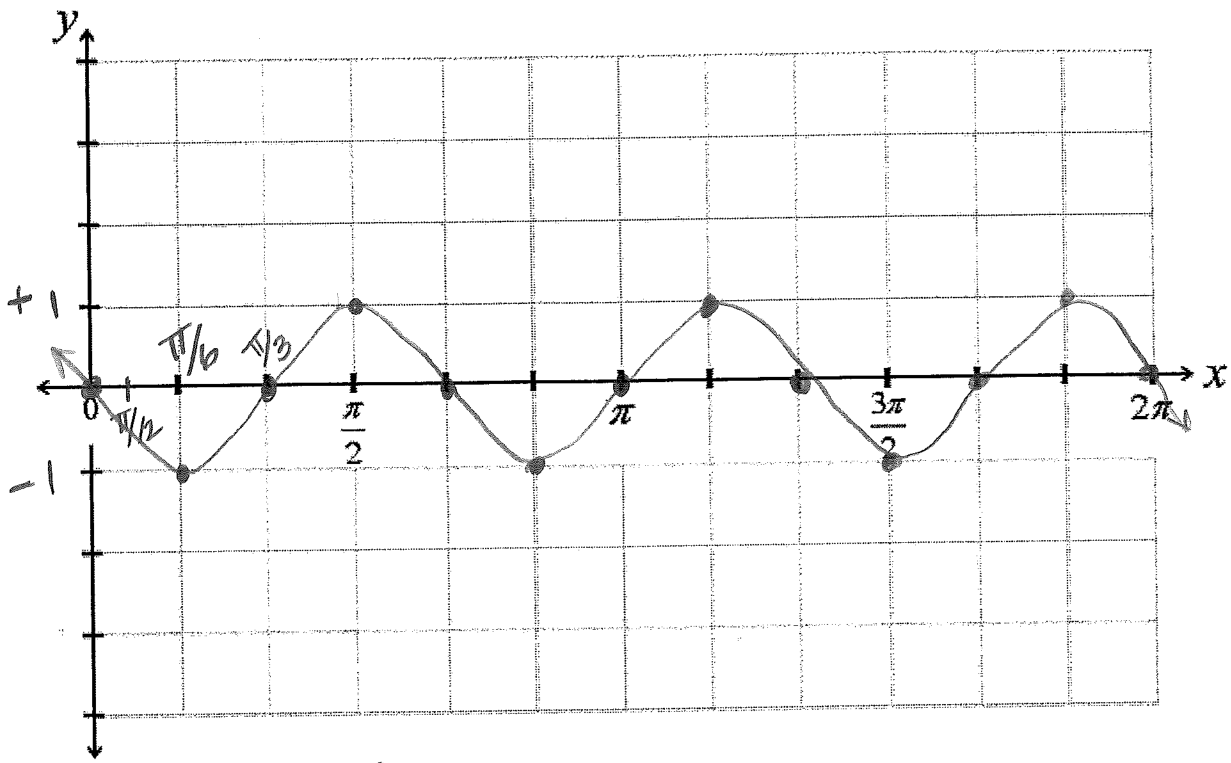
$\frac{3\pi}{6} + \frac{\pi}{6} = \frac{4\pi}{6}$



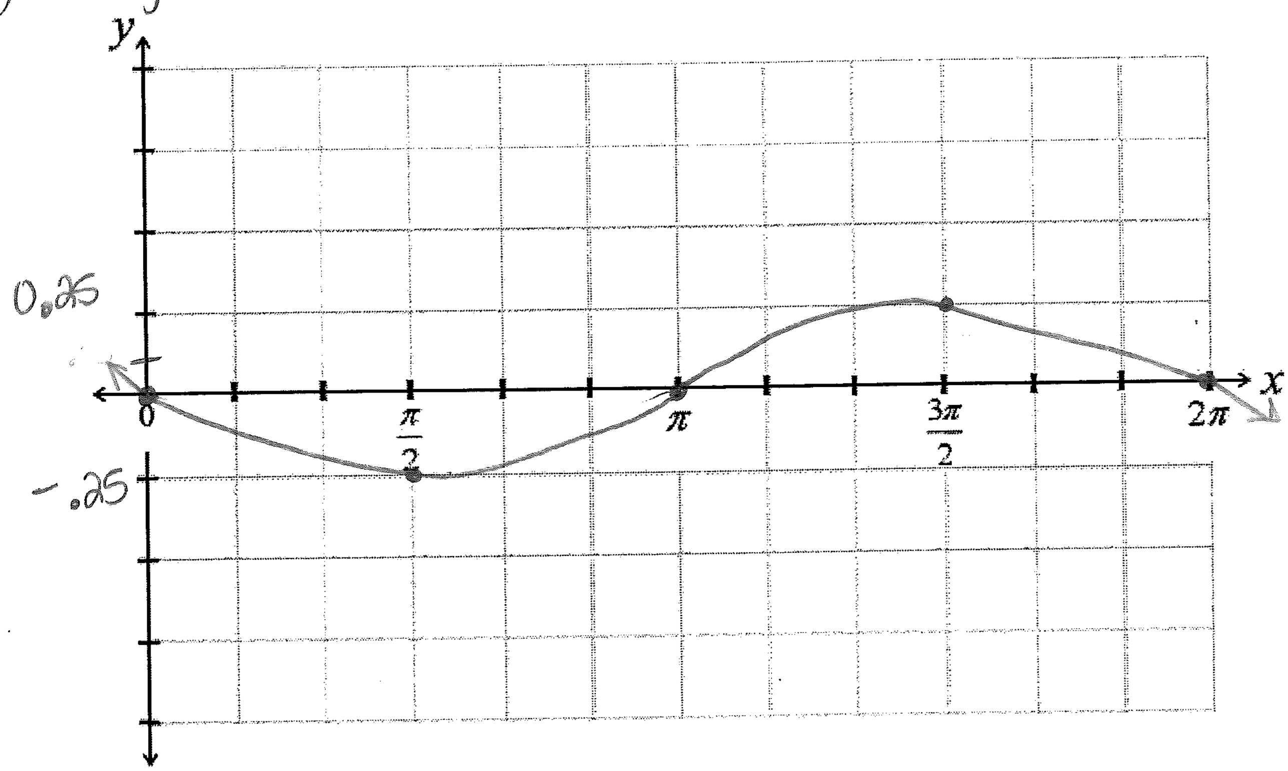
⑩ $y = -2\sin \frac{1}{2}\theta$



(11) $y = -\sin 3x$



(12) $y = -\frac{1}{4}\sin\theta$



Name Key Class ____________ Date ____________

Key

5-4 Notes

The Cosine Function

*graph paper for cos function

Problem

What is the graph of $y = 3\cos\frac{\pi}{2}\theta$ in the interval from 0 to 2π?

Step 1 Compare the function to $y = a \cos b\theta$. $a = 3$ and $b = \frac{\pi}{2}$

Find the amplitude. $|a| = |3| = 3$

Find the period of the curve. $\frac{2\pi}{b} = \frac{2\pi}{\frac{\pi}{2}} = 4$

Step 2 Find the minimum and maximum of the curve.
Because the amplitude is 3, the maximum is 3 and the minimum is –3.

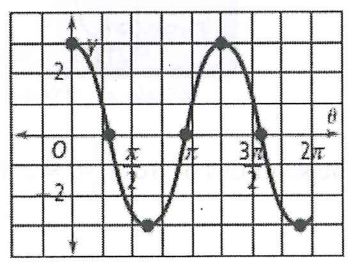
Step 3 Make a table of values. Choose θ-values at intervals of one-fourth the period: $\frac{4}{4} = 1$.

The y-values cycle through the pattern *max-zero-min-zero-max*.

θ	0	1	2	3	4	5	6
y	3	0	–3	0	3	0	–3

Step 4 Plot the points from the table.

Step 5 Draw a smooth curve through the points.



*zoom stat.

Exercises

Sketch the graph of each function in the interval from 0 to 2π.

1. $y = \frac{1}{2}\cos 2\theta$

amp: 1/2
period: π

2. $y = 3\cos\frac{1}{2}\theta$

amp: 3
p: 4π

3. $y = \cos 3\theta$

amp: 1
p = 2π/3

4. $y = \frac{1}{4}\cos\pi\theta$

A = 1/4
period: 2

5. $y = -2\cos\frac{1}{2}\theta$

am: 2
p: 4π

6. $y = 2\cos 6\pi\theta$

don't graph

Name ________________________________ Class ____________ Date ______________

5-4 Notes (continued)

The Cosine Function

Solving a sine or cosine equation is similar to solving a system of two linear equations. You can graph each side of the equation. The solutions will be the points where the graphs intersect.

Problem

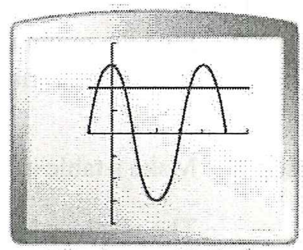
What are the solutions of $3\cos\frac{1}{2}\theta = 2$ in the interval 0 to 4π? — change window

Step 1 Set each side of the equation equal to y.

$$y = 3\cos\frac{1}{2}\theta$$

$$y = 2$$

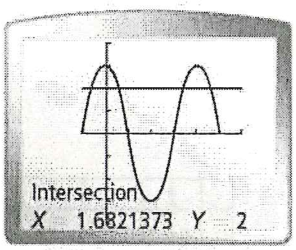
Step 2 Graph each equation on the same grid.



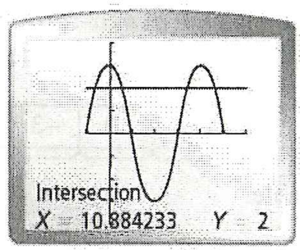
X Scale: π y Scale: 1

Step 3 Between $\theta = 0$ and $\theta = 4\pi$, the graphs intersect 2 times. Use the **Intersect** feature to find the coordinates of these points.

*Zoom Stat
window:
X-scale π
X-max: 4π



X Scale: π Y Scale: 1



X Scale: π Y Scale: 1

The solutions of $3\cos\frac{1}{2}\theta = 2$ in the interval 0 to 4π are $\theta \approx 1.68$ and 10.88.

Exercises

Set window

Find all solutions in the interval from 0 to 2π. Round to the nearest hundredth. (Radian mode)

7. $-\cos\theta = \frac{3}{4}$

$\theta = 2.42, 3.864$

8. $2\cos\theta = 1$

$\theta = 1.05, 5.24$

9. $3\cos\pi\theta = 2$

10. $\cos\frac{1}{2}\pi\theta = -0.5$

1.33, 2.67, 5.33

11. $\frac{1}{2}\cos 4\theta = 0$ → x-axis

0.39, 1.18, 1.96, 2.75, 3.53, 4.32, 5.11, 5.89

12. $-3\cos 2\pi\theta = 2.5$

13. $5\cos 4\theta = 3$

14. $\frac{3}{4}\cos\frac{1}{2}\pi\theta = \frac{1}{2}$

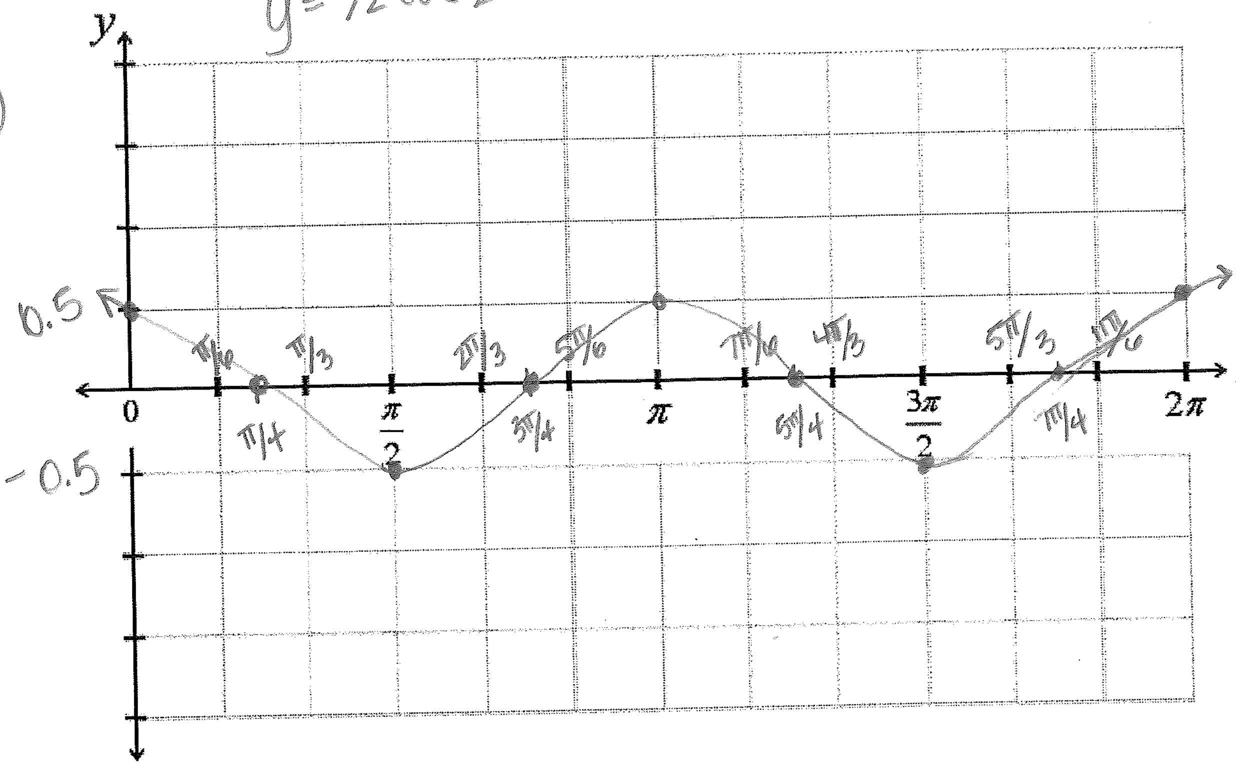
0.54, 3.46, 4.54

15. $-4\cos 2\theta = 2$

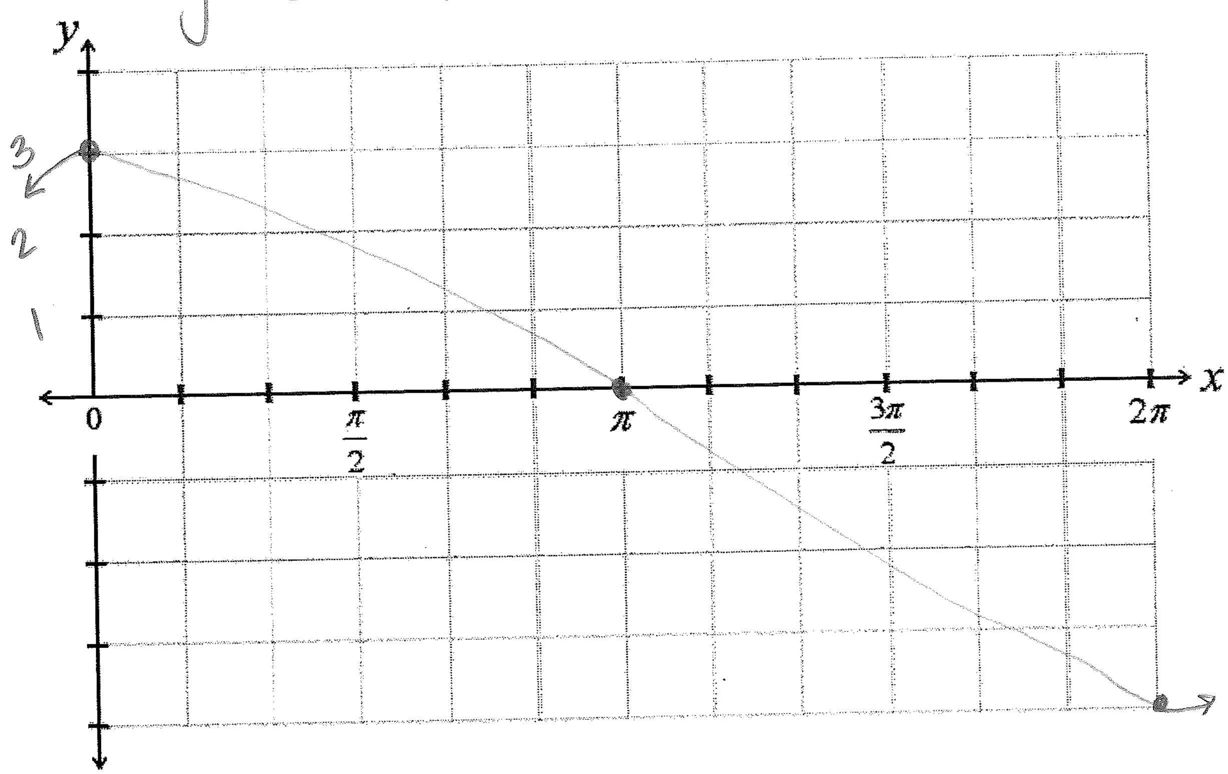
1.05, 2.09, 4.19, 5.24

5-4 Notes - The Cosine Function

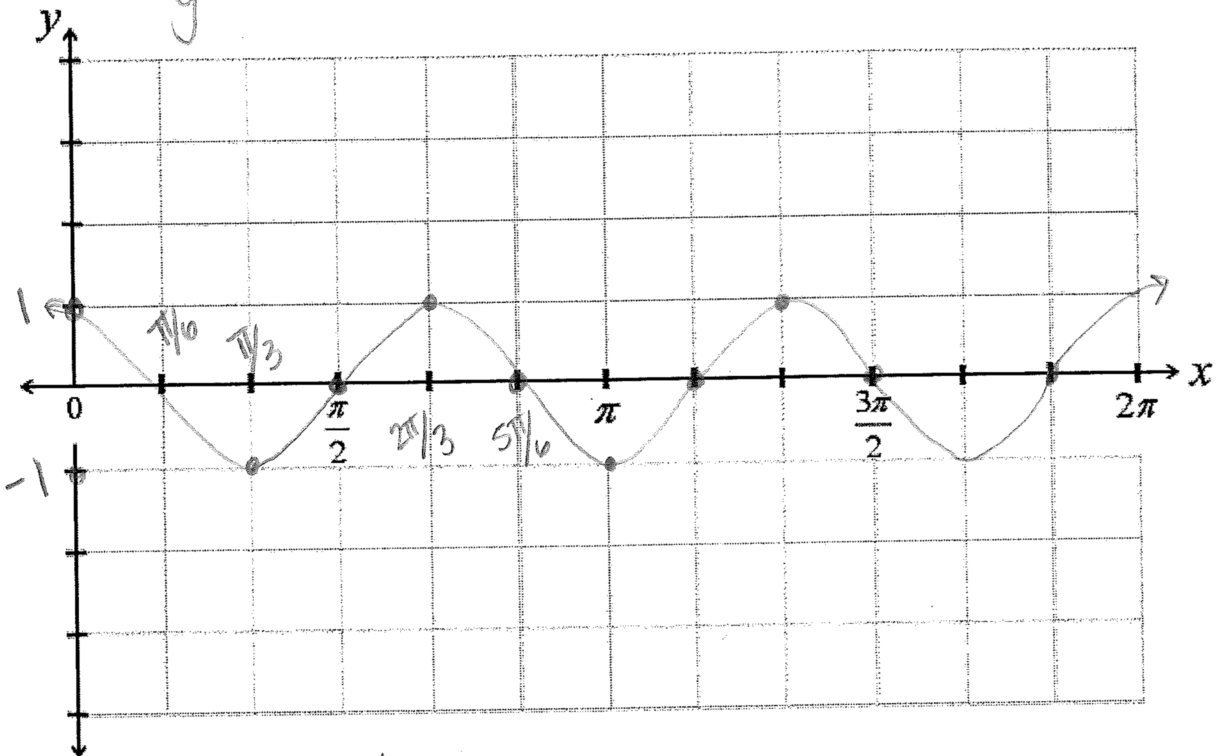
1. $y = \frac{1}{2}\cos 2\theta$



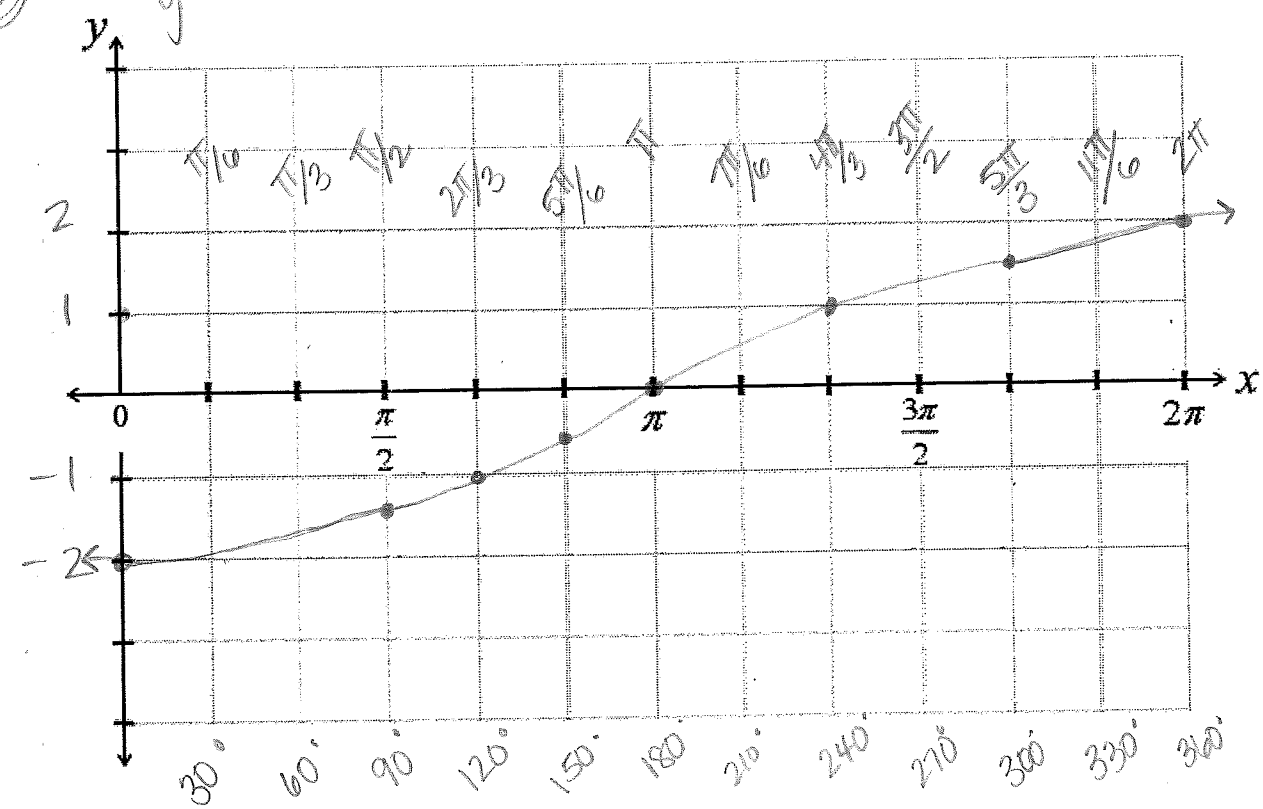
2. $y = 3\cos \frac{1}{2}\theta$



(3) $y = \cos 3\theta$



(5) $y = -2\cos \frac{1}{2}\theta$



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Math III
Notes 5-6 Translating Sine and Cosine Functions

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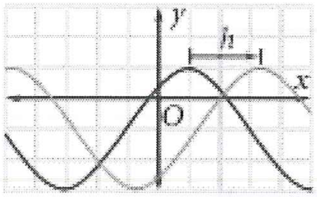
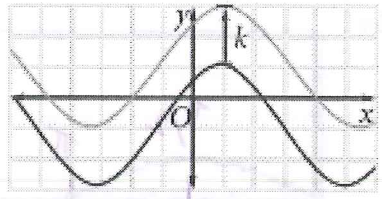
You can translate periodic functions horizontally and vertically using the methods you have used for other functions.



$g(x)$: horizontal translation of $f(x)$
$g(x) = f(x - h)$



$h(x)$: vertical translation of $f(x)$
$h(x) = f(x) + k$

Each horizontal translation of certain periodic functions is a phase shift.

When $g(x) = f(x-h)$, the value of h is the amount of the shift left or right.

If $h > 0$, the shift is right. If $h < 0$, the shift is left.

Identifying Phase Shifts

What is the value of h in each translation? Describe each phase shift using a phrase such as 3 *units to the left.*

a. $g(x) = f(x-2)$

$h = 2$

2 units to the right

b. $y = \cos(x+4)$

$h = -4$

$x - h$

$x - (-4)$

$x + 4$

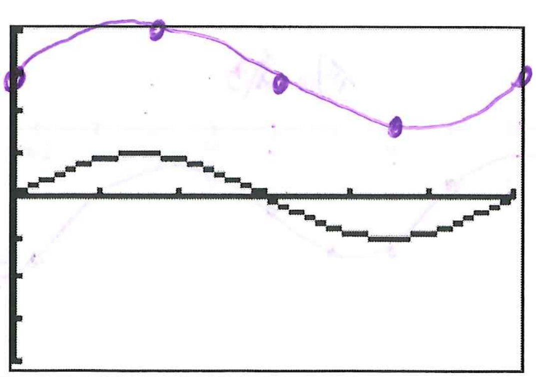
4 units left

Graphing Translations

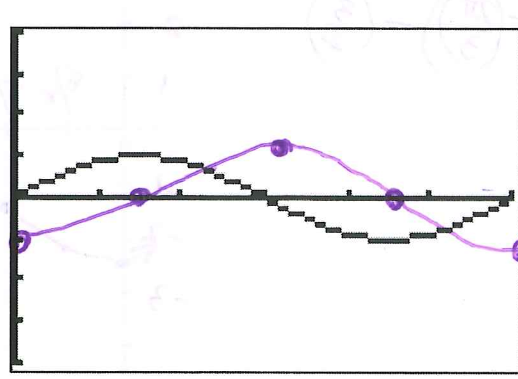
Use the graph of the parent function $y = \sin x$. Sketch each translation of the graph in the interval $0 \le x \le 2\pi$.

Radian mode

a. $y = \sin x + 3$



b. $y = \sin\left(x - \dfrac{\pi}{2}\right)$



Math III
Notes 5-6 Translating Sine and Cosine Functions

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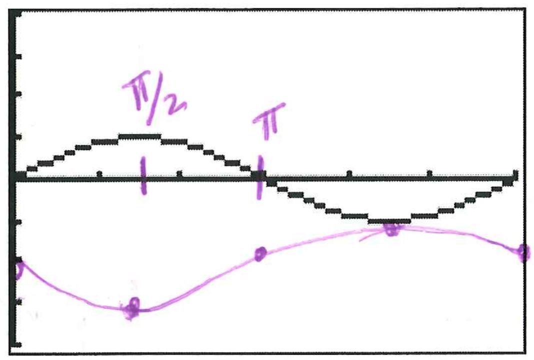
(2)

Graphing a Combined Translation

radian mode

Using the graph of the parent function $y=\sin x$, sketch the translation $y=\sin(x+\pi)-2$ in the interval $0 \le x \le 2\pi$.



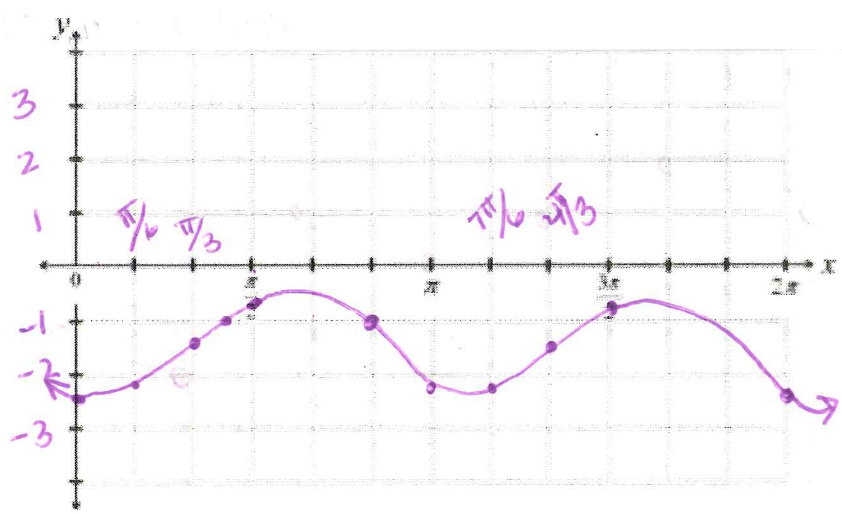
Summary – Families of Sine and Cosine Functions

Parent Function	Transformed Function
$y=\sin x$	$y=a\sin b(x-h)+k$
$y=\cos x$	$y=a\cos b(x-h)+k$

- $|a|$ = amplitude (vertical stretch or shrink)
- $\frac{2\pi}{b}$ = period (when x is in radians and $b>0$)
- h = phase shift, or horizontal shift
- k = vertical shift

Graph $y=\sin 2\left(x-\frac{\pi}{3}\right)-\frac{3}{2}$

$y=\sin\left(2x-\left(\frac{2\pi}{3}\right)\right)-\left(\frac{3}{2}\right)$



Math III
Notes 5-7 Trigonometric Identities

Name______________________
Date______________ Period ________

Trigonometric Identities

A trigonometric function is a trigonometric equation that is true for all values except those for which the expressions of either side of the equal sign are undefined.

Reciprocal Identities

The cosecant (csc), secant (sec), and cotangent (cot) functions are defined as reciprocals. Their domains include all real numbers Θ except those that make a denominator zero.

$\csc\Theta = \frac{1}{\sin\theta}$	$\sec\Theta = \frac{1}{\cos\theta}$	$\cot\Theta = \frac{1}{\tan\theta}$

Using Reciprocals

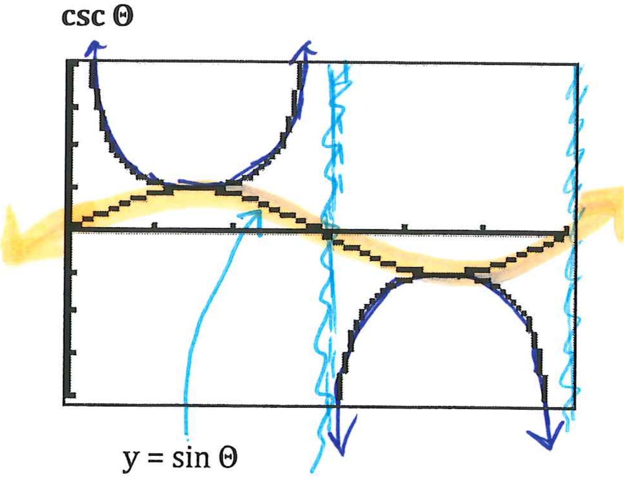
(Degree mode)

a. Find csc 60° $\frac{1}{\sin 60} = 1.154$	b. Suppose $\cos\Theta = \frac{5}{13}$. Find sec Θ. $\sec\theta = \frac{1}{\cos\theta} \Rightarrow \frac{1}{5/13} \Rightarrow 1 \cdot \frac{13}{5}$

$= \frac{13}{5}$

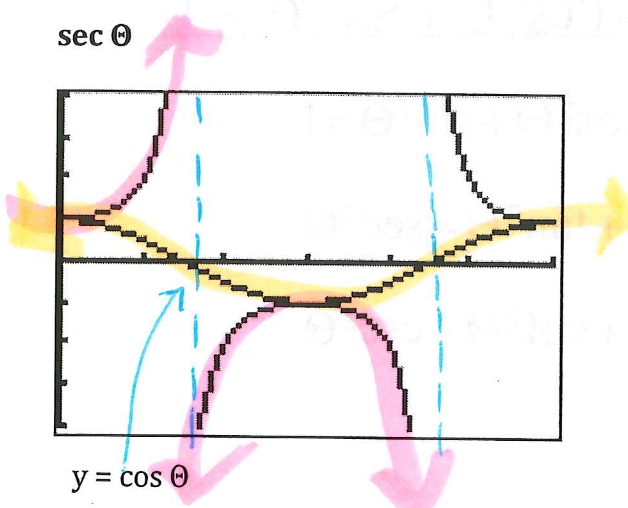
Graphs of Reciprocal Trig Functions

csc Θ



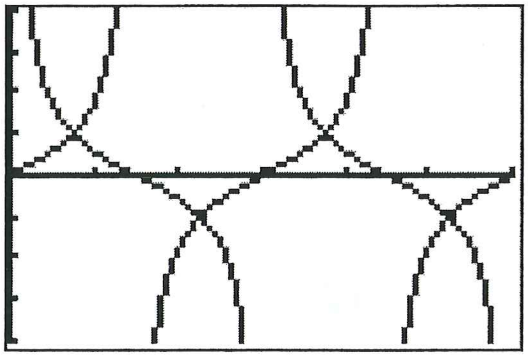
y = sin Θ

sec Θ



y = cos Θ

cot Θ



y = tan Θ

Tangent and Cotangent Identities

$$\tan\Theta = \frac{\sin\Theta}{\cos\Theta} \qquad \cot\Theta = \frac{\cos\Theta}{\sin\Theta}$$

$$\frac{1}{\tan\theta} = \frac{1}{\left(\frac{\sin\theta}{\cos\theta}\right)} = \frac{\cos\theta}{\sin\theta}$$

Pythagorean Identities

You can derive another identity from the definitions of cos Θ and sin Θ.

The ordered pair $(\cos\theta, \sin\theta)$ is a point on the unit circle, and for any point

(x, y) on the unit circle $x^2 + y^2 = 1^2$.

So $\cos^2\theta + \sin^2\theta = 1$.

$$\cos^2\Theta + \sin^2\Theta = 1$$

$$1 + \tan^2\Theta = \sec^2\Theta$$

$$1 + \cot^2\Theta = \csc^2\Theta$$